Junior Division: Problems 1 solutions

J1. Twenty-seven unit cubes are each coloured completely blue or completely yellow. All twenty-seven unit cubes are assembled into a larger cube. If half of the surface area of the larger cube is blue, what is the largest number of unit cubes that could have been coloured blue?

Solution

The surface area of one face of a 3 x 3 unit cube is 9 square units.

There are 6 faces, so 54 square units in all.

The largest number of blue cubes occurs with the smallest number of yellow cubes, and it's easier to count fewer cubes.

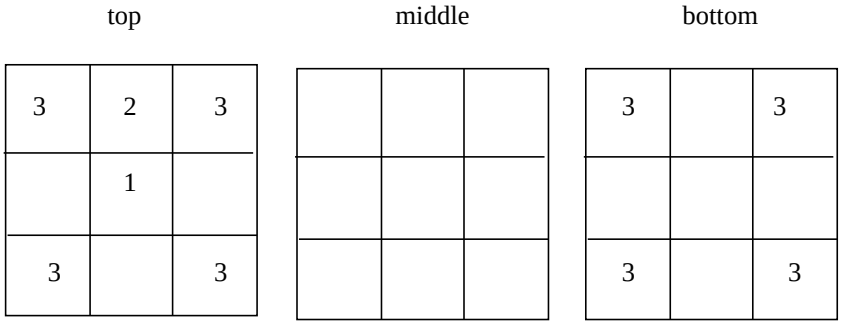
One half of the surface area is yellow, i.e. $\frac{1}{2} \times 54 = 27$ unit faces.

The greatest area of yellow from a unit yellow cube is when it is in a corner of the large cube, which would supply 3 unit faces. We require 27 unit faces, so putting yellow cubes in all 8 corners will account for 24 of the yellow unit faces.

The next greatest area of yellow from a unit yellow cube is when it is in the middle of an edge of the large cube, which would supply 2 unit faces. So we need one edge unit cube to take the total to 26 yellow unit faces.

The final unit face comes from a yellow cube in the centre of a face of the large cube. This gives 10 yellow cubes.

(The numbers in the diagram below show how many faces of a yellow cube can be seen.)



A corner yellow cube and the yellow cube in the centre of a face could be replaced by 2 edge yellow cubes, but that is still a total of 10 yellow cubes.

So there is a minimum of 10 unit cubes coloured yellow and hence a maximum of $27 - 10 = 17$ unit cubes coloured blue.

Alternative solution

The surface area of one face of a 3×3 unit cube is 9 square units.

There are 6 faces, so 54 square units in all.

One half of the surface area is blue, i.e. 27 unit faces.

First make the hidden central unit cube blue.

Now consider the visible unit cubes.

The least area of blue from a unit blue cube is when it is in the centre of a face of the large cube, which would supply 1 unit face. Putting blue cubes in all 6 centre faces will account for 6 of the blue unit faces.

The next smallest area of blue from a unit blue cube is when it is in the middle of an edge of the large cube, which would supply 2 unit faces. All 12 edge unit cubes would take the total to $2 \times 12 + 6 = 30$ blue unit faces. But the target is 27 blue faces, so remove 3 edge unit cubes and replace with one corner unit cube to get 27 blue faces exactly. This uses $6 + 9 + 1 = 16$ blue unit cubes. Or remove 1 centre cube and 1 edge cube, which uses $5 + 11 = 16$ blue unit cubes.

Either way, the largest possible number of blue unit cubes is $1 + 16 = 17$.

J2. One morning in the staff room, the coffee jar was empty! The headteacher asks the four teachers who used the last of the coffee?

Andrew: “It wasn’t me”

Brian: “It wasn’t me”

Carol: “It was Diane”

Diane: “It was Brian”

It turns out exactly one of them did not tell the truth – who used the last of the coffee?

Solution

If Andrew did not tell the truth and everybody else did, then

A: It was Andrew

B: It wasn’t Brian

C: It was Diane – this causes a contradiction

If Brian did not tell the truth then similarly,

A: It wasn’t Andrew

B: It was Brian

C: It was Diane – contradiction

If Carol did not the truth then,

A: it wasn’t Andrew

B: It wasn’t Brian

C: It wasn’t Diane

D: It was Brian - contradiction

If Diane was the one who did not tell the truth then,

A: it wasn’t Andrew

B: It wasn’t Brian

C: It was Diane

D: It wasn’t Brian

The culprit was Diane.

J3. An unusual type of die in the form of a cuboctahedron has 6 identical square faces and 8 identical equilateral triangular faces. It is twice as likely to land on a square face as on a triangular face. What is the probability that when the die is thrown that it lands on a triangular face?

Solution

Let the probability of landing on a triangular face be p.

Then $8p + 6 \times 2p = 1$,

and so $p = \frac{1}{20}$.

The probability of landing on one of the 8 triangular faces is $8p = \frac{8}{20} = \frac{2}{5}$.

J4. On one type of car, a Goodrock tyre lasts for 20,000 miles on a front wheel or 30,000 miles on a rear wheel. By interchanging the front and rear tyres, what is the greatest distance that can be driven on one set of four tyres?

What is the most efficient way to interchange the tyres?

Solution

Let the maximum distance be x miles.

Each tyre must travel $\frac{1}{2}x$ miles on a front wheel and $\frac{1}{2}x$ miles on a rear wheel, otherwise the car would not have a full complement of tyres at all times.

The proportion of the tyre life used is $\frac{\frac{1}{2}x}{20000}$ on the front and $\frac{\frac{1}{2}x}{30000}$ on the rear.

After x miles the whole of the tyre life is used, so

$$\frac{\frac{1}{2}x}{20000} + \frac{\frac{1}{2}x}{30000} = 1$$

Hence $x = 24000$.

The maximum distance on one set of tyres is 24000 miles.

Only one change is needed: interchange the front and rear tyres after 12000 miles.

(Each tyre uses 3/5 of its life on the front (3/5 of 20000 = 12000 miles) and 2/5 of its life on the rear (2/5 of 30000 = 12000 miles).)

J5. During training, five ladies of the Lightweight Rowing Team went on the scales two at a time and every possible combination of two ladies was recorded which resulted in 10 separate weighings. The 10 weighings were as follows:

110 kg, 112 kg, 113 kg, 114 kg, 115 kg, 116 kg, 117 kg, 118 kg, 120 kg, 121 kg.

What was the weight of each lady?

Solution

In the 10 weighings, each lady is weighed 4 times.

So the sum of the 10 weighings will be 4 times the sum of the five ladies weights.

The sum is 1156 kg so the sum of the 5 ladies weights is 289 kg.

Since the lightest two weigh 110 kg together and the heaviest two weigh 121 kg together, the middle one will weigh 289 – (110 + 121) = 58 kg.

But the lightest one and the middle one together weigh 112 kg and so the lightest weighs 112 – 58 = 54 kg.

So the second lightest weighs 110 – 54 = 56 kg.

In the same way, the heaviest and the middle weigh 120 kg so the heaviest weighs 120 – 58 = 62 kg. Then the second heaviest weighs 121 – 62 = 59 kg.

So the weights are 54 kg, 56 kg, 58 kg, 59 kg and 62 kg.